

ON THE POSITIVE COMMUTATOR IN THE RADICAL

MARKO KANDIĆ, KLEMEN ŠIVIC

ABSTRACT. In this paper we prove that a positive commutator between a positive compact operator A and a positive operator B is in the radical of the Banach algebra generated by A and B . Furthermore, on every at least three-dimensional Banach lattice we construct finite rank operators A and B satisfying $AB \geq BA \geq 0$ such that the commutator $AB - BA$ is not contained in the radical of the Banach algebra generated by A and B . These two results now completely answer to two open questions published in [4]. We also obtain relevant results in the case of the Volterra and the Donoghue operator.

Math. Subj. Classification (2010): 47A10; 47A46; 47B07; 47B47.

Key words: Positive commutators, Positive compact operators, Invariant subspaces, Radical.

1. INTRODUCTION

Positive commutators of positive operators on Banach lattices were first considered in [4]. The authors proved [4, Theorem 2.2] that a positive commutator $AB - BA$ of positive compact operators A and B is necessarily quasi-nilpotent and furthermore, it is contained in the radical of the Banach algebra generated by A and B . They posed a question if the same is still true if we assume that only one of the operators A and B is compact. The first part of the problem was independently solved by Drnovšek [5] and Gao [10]. Drnovšek proved even more.

Theorem 1.1. [5, Theorem 3.1] *Let A and B be bounded operators on a Banach lattice E such that $AB \geq BA \geq 0$ and AB is power-compact. Then $AB - BA$ is an ideal-triangularizable quasi-nilpotent operator.*

This result is an improvement of [4, Theorem 2.4] where the authors considered operators from appropriate Schatten ideals. In this paper we prove that a positive commutator $AB - BA$ of positive operators A and B is in the radical of the Banach algebra generated by A and B whenever one of the operators A and B is compact. We actually prove that $AB - BA$ is contained in the radical of the possibly bigger algebra that contains the Banach algebra generated by A and B . This result now completely solves the open problem [4, Open questions 3.7 (1)]. We also prove that on every at least three-dimensional Banach lattice E there exist finite rank operators A and B on E satisfying $AB \geq BA \geq 0$, however the commutator $AB - BA$ is not contained in the radical of the Banach algebra generated by A and B . Although this counterexample answers negatively to [4, Open questions 3.7(2)] in the case of at least three-dimensional Banach lattices, there is a positive answer on two-dimensional Banach lattices.

2. PRELIMINARIES

Let E be a Riesz space, and let E^+ denote the set of all positive vectors in E . A linear subspace J of E is said to be an *ideal* whenever $0 \leq |x| \leq |y|$ and $y \in J$ imply $x \in J$. An order closed ideal is said to be a *band*. The set

$$A^d := \{x \in E : |x| \wedge |a| = 0 \text{ for all } a \in A\}$$

is called the *disjoint complement* of a set A in E . The disjoint complement of a nonempty set in E is always a band in E . A band B of E is said to be a *projection band* if $E = B \oplus B^d$. A Riesz space E is said to be a *normed Riesz space* whenever E is equipped with a lattice norm. If a normed Riesz space is a complete metric space with the metric induced by the norm, then it is called a *Banach lattice*. It is well-known that every normed Riesz space is Archimedean.

A nonzero vector $a \in E^+$ is an *atom* in a normed Riesz space E if $0 \leq x, y \leq a$ and $x \wedge y = 0$ imply either $x = 0$ or $y = 0$, or equivalently, if $0 \leq x \leq a$ implies $x = \lambda a$ for some $\lambda \geq 0$, i.e., the principal ideal B_a generated by a is one-dimensional. It turns out that B_a is a projection band [13]. If every set of mutually disjoint nonzero vectors of an Archimedean Riesz space E is finite, then E is a finite-dimensional atomic lattice that is order isomorphic to \mathbb{R}^n (where $n = \dim E$) with componentwise ordering. If an

Archimedean Riesz space E contains only finitely many pairwise disjoint atoms, then E is an order direct sum of the finite-dimensional atomic part of the lattice E and its disjoint complement that is an atomless part of the lattice E .

An operator T on a Banach lattice E is said to be positive whenever T maps the positive cone of E into itself. It is well-known that every positive operator on a Banach lattice is a bounded operator. By $\mathcal{L}(X)$ and $\mathcal{L}_+(E)$ we denote the Banach algebra of all bounded operators on a Banach space X and the set of all positive operators on a Banach lattice E , respectively. A family \mathcal{F} of operators on a Banach space X is *reducible* if there exists a nontrivial closed subspace of X that is invariant under every operator from \mathcal{F} . If there exists a maximal chain \mathcal{C} of closed subspaces of X such that every subspace from the chain \mathcal{C} is invariant under every operator from \mathcal{F} , then \mathcal{F} is said to be *triangularizable*, and \mathcal{C} is called a *triangularizing chain* for \mathcal{F} . A family \mathcal{F} of operators on a Banach lattice E is said to be *ideal-reducible* if there exists a nontrivial closed ideal of E that is invariant under every operator from \mathcal{F} . Otherwise, we say that \mathcal{F} is *ideal-irreducible*. A family \mathcal{F} of operators on a Banach lattice E is said to be *ideal-triangularizable* whenever there exists a chain of closed ideals of E that is maximal as a chain in the lattice of all closed ideals of E and every ideal from the chain is invariant under \mathcal{F} . Every ideal-triangularizable family of operators is also triangularizable [6]. If \mathcal{C} is a complete chain of closed subspaces of a Banach space X , the *predecessor* of \mathcal{M} in \mathcal{C} is denoted by \mathcal{M}_- . If every subspace \mathcal{M} from a complete chain \mathcal{C} of closed subspaces of X is invariant under the operator $A \in \mathcal{L}(X)$ and $\mathcal{M} \neq \mathcal{M}_-$, the induced operator $A_{\mathcal{M}}$ on $\mathcal{M}/\mathcal{M}_-$ is defined by $A_{\mathcal{M}}(x + \mathcal{M}_-) = Ax + \mathcal{M}_-$ for all $x \in \mathcal{M}$. In the case of a finite-dimensional lattice \mathbb{R}^n we will use more common terms decomposable, indecomposable and completely decomposable instead of ideal-reducible, ideal-irreducible and ideal-triangularizable, respectively. For a more detailed treatment on triangularizability we refer the reader to [15].

Positive operators A and B on a Banach lattice *semi-commute* whenever $AB \geq BA$ or $AB \leq BA$. The *super right-commutant* $[A]$ and the *super left-commutant* $\langle A \rangle$ of a positive operator A on a Banach lattice E are defined by

$$[A] = \{B \in \mathcal{L}_+(E) : BA \geq AB\} \quad \text{and} \quad \langle A \rangle = \{B \in \mathcal{L}_+(E) : AB \geq BA\},$$

respectively. For the terminology and details not explained about Banach lattices and operators on them we refer the reader to [2] and [16].

The *radical* of a Banach algebra \mathcal{A} is the intersection of all maximal modular left ideals of \mathcal{A} which coincides with the intersection of all maximal modular right ideals of \mathcal{A} . If \mathcal{A} is a unital Banach algebra, then the radical of \mathcal{A} is equal to the set

$$\{a \in \mathcal{A} : r(ba) = 0 \text{ for all } b \in \mathcal{A}\} = \{a \in \mathcal{A} : r(ab) = 0 \text{ for all } b \in \mathcal{A}\}$$

where $r(x)$ denotes the spectral radius of an element $x \in \mathcal{A}$. Therefore, the radical of a unital Banach algebra is the largest among all ideals that consist of quasi-nilpotent elements. If a Banach algebra \mathcal{A} is not unital, then we consider the spectrum of $a \in \mathcal{A}$ as the spectrum of the element a in the standard unitization of the algebra \mathcal{A} . For the terminology not explained throughout the text regarding Banach algebras we refer the reader to [3].

3. COUNTEREXAMPLES

In [4] the authors proved [4, Theorem 2.3] that the commutator $AB - BA$ of real matrices A and B is a nilpotent matrix whenever $AB \geq BA \geq 0$. The authors also extended this result to the case when $A \in \mathcal{C}^p$ and $B \in \mathcal{C}^q$ are bounded operators on l^2 with $AB \geq BA \geq 0$ where $1 \leq p, q \leq \infty$ and $1/p + 1/q = 1$. In this case, the commutator is a quasi-nilpotent trace class operator on l^2 . See [4, Theorem 2.4]. They asked ([4, Open questions 3.7(2)]) if the commutator $AB - BA$ is contained in the radical of the Banach algebra generated by A and B . In this section we provide a counterexample on every at least three-dimensional Banach lattice.

For the construction of our counterexample to [4, Open questions 3.7(2)] we need some preparation. The proof of the following result can be found in [13, Theorem 26.10]. We provide our proof for the sake of completeness.

Proposition 3.1. *Every infinite-dimensional Banach lattice has at least countably infinitely many nonzero pairwise disjoint positive vectors.*

Proof. Let us choose an arbitrary infinite-dimensional Banach lattice E . Suppose first that E does not contain atoms. We claim that there exists an increasing sequence of sets $\{\mathcal{A}_n\}_{n \in \mathbb{N}}$ of positive nonzero pairwise disjoint vectors with $|\mathcal{A}_n| = n$ and a sequence of

positive nonzero vectors $\{y_n\}_{n \in \mathbb{N}}$ such that y_n is disjoint with \mathcal{A}_n for every positive integer n . Let us validate the statement above when $n = 1$. If y_0 is an arbitrary nonzero positive vector in E , then the fact that E is without atoms implies that there exist nonzero positive vectors $0 \leq x_1, y_1 \leq y_0$ satisfying $x_1 \wedge y_1 = 0$. We choose $\mathcal{A}_1 = \{x_1\}$ and y_1 is disjoint with \mathcal{A}_1 . Suppose that there exist sets $\mathcal{A}_1 \subseteq \cdots \subseteq \mathcal{A}_n$ of positive nonzero pairwise disjoint vectors with $|\mathcal{A}_j| = j$ for $1 \leq j \leq n$, and positive nonzero vectors y_1, \dots, y_n such that $y_j \in \mathcal{A}_j^d$ for $1 \leq j \leq n$. Since y_n is not atom, there exist positive nonzero vectors x_{n+1} and y_{n+1} satisfying $0 \leq x_{n+1}, y_{n+1} \leq y_n$ and $x_{n+1} \wedge y_{n+1} = 0$. It is obvious that the vectors x_{n+1} and y_{n+1} are disjoint with \mathcal{A}_n . The set

$$\mathcal{A}_{n+1} := \mathcal{A}_n \cup \{x_{n+1}\}$$

obviously satisfies $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$, $|\mathcal{A}_{n+1}| = n + 1$ and $y_{n+1} \in \mathcal{A}_{n+1}^d$.

Let $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$. The set \mathcal{A} is countably infinite, and arbitrary two vectors from \mathcal{A} are disjoint.

Suppose now that E contains at least one atom. By Zorn's lemma there exists a maximal set \mathcal{A} of pairwise disjoint atoms of norm one. If \mathcal{A} is infinite, then we are done. Suppose otherwise that \mathcal{A} is a finite set, so that we have $E = \mathcal{A}^{dd} \oplus \mathcal{A}^d$ where \mathcal{A}^{dd} is a finite-dimensional atomic lattice and \mathcal{A}^d is an infinite-dimensional atomless lattice. By already proved, \mathcal{A}^d contains at least countably infinitely many nonzero positive pairwise disjoint vectors. \square

Lemma 3.2. *Let E be a Banach lattice of dimension at least n and let x_1, \dots, x_n be nonzero positive pairwise disjoint vectors. Then there exist nonzero positive functionals $\varphi_1, \dots, \varphi_n$ on E such that $\varphi_i(x_j) = \delta_{ij}$ for $i, j = 1, \dots, n$.*

Proof. Let us choose an arbitrary $1 \leq i \leq n$ and let \mathcal{J}_i be the ideal generated by the vector x_i . By [18, Theorem 39.3.], there exists a positive functional ψ_i on E with $\psi_i(x_i) = 1$. Let $\psi_i|_{\mathcal{J}_i}$ be the restriction of the functional ψ_i on the ideal \mathcal{J}_i . By [18, Theorem 20.5.] and its following remark there exists a positive functional φ_i on E which extends $\psi_i|_{\mathcal{J}_i}$ and φ_i is zero on \mathcal{J}_i^d . This implies $\varphi_i(x_j) = \delta_{ij}$ for all $j = 1, \dots, n$. \square

Although Lemma 3.2 is crucial for Example 3.6 and Example 3.7, it is also of its own interest. Before we indicate its importance let us recall first the well-known Jacobson's density theorem for Banach algebras [3].

Theorem 3.3. *Let \mathcal{A} be a Banach algebra and let $\pi : \mathcal{A} \rightarrow \mathcal{L}(X)$ be a continuous irreducible representation on a Banach space X . If x_1, \dots, x_n are linearly independent in X and y_1, \dots, y_n are arbitrary in X , then there exists $a \in \mathcal{A}$ such that $\pi(a)x_i = y_i$ for all $i = 1, \dots, n$.*

In the special case when π is the identity representation on $\mathcal{L}(X)$ Jacobson's density theorem states that a finite set of linearly independent vectors can be mapped onto arbitrary vectors by bounded operators. This fact can be also proved with an application of the Hahn-Banach theorem without applying the Jacobson's density theorem. Now it is natural to ask whether arbitrary linearly independent positive vectors in Banach lattices can be mapped to arbitrary positive vectors by positive operators. The following example shows that in general it cannot be done.

Example 3.4. *The space \mathbb{R}^2 with a canonical ordering and equipped with the norm $\|\cdot\|_2$ is a Banach lattice. The only matrix mapping $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is*

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

which is clearly not positive.

However, for a finite set of nonzero pairwise disjoint positive vectors we have a positive result.

Theorem 3.5. *Let E be a Banach lattice and let x_1, \dots, x_n be arbitrary nonzero positive pairwise disjoint vectors in E . Then for arbitrary positive vectors y_1, \dots, y_n in E there exists a positive operator T on E such that $Tx_j = y_j$ for all $j = 1, \dots, n$.*

Proof. By Lemma 3.2 there exist positive functionals $\varphi_1, \dots, \varphi_n$ on E such that $\varphi_i(x_j) = \delta_{ij}$ for all $i, j = 1, \dots, n$. The positive operator

$$T = \sum_{i=1}^n y_i \otimes \varphi_i$$

satisfies

$$Tx_j = \sum_{i=1}^n \varphi_i(x_j)y_i = \sum_{i=1}^n \delta_{ij}y_j = y_j$$

for all $j = 1, \dots, n$. □

Throughout the rest of this section E is assumed to be at least three-dimensional Banach lattice. Let the vectors x_1, x_2 and x_3 and the functionals φ_1, φ_2 and φ_3 be as in Lemma 3.2. The vectors x_1, x_2 and x_3 , in fact, exist since either E is finite-dimensional and so it is order isomorphic to \mathbb{R}^n with the componentwise order, or E is infinite-dimensional and we may apply Proposition 3.1.

The following example shows that the commutator $AB - BA$ of compact operators A and B with $AB \geq BA \geq 0$ is in general not contained in the radical of the Banach algebra generated by A and B if we do not assume $A \geq 0$.

Example 3.6. *An easy calculation shows that operators*

$$A = (-x_2 + x_3) \otimes \varphi_1 + (x_1 + x_2) \otimes \varphi_2 \quad \text{and} \quad B = x_2 \otimes (\varphi_2 + \varphi_3)$$

satisfy

$$AB = (x_1 + x_2) \otimes (\varphi_2 + \varphi_3) \quad \text{and} \quad BA = x_2 \otimes \varphi_2,$$

so that AB and BA are positive operators on E . Since

$$AB - BA = x_1 \otimes (\varphi_2 + \varphi_3) + x_2 \otimes \varphi_3 \geq 0,$$

we also have $AB \geq BA \geq 0$. The commutator $AB - BA$ is not contained in the radical of the Banach algebra \mathcal{A} generated by the operators A and B since the operator

$$A(AB - BA) = (x_3 - x_2) \otimes \varphi_2 + (x_1 + x_3) \otimes \varphi_3$$

has 1 as its eigenvalue with a corresponding eigenvector $x_1 + x_3$.

The following example shows that the commutator $AB - BA$ of compact operators A and B with $AB \geq BA \geq 0$ is in general not contained in the radical of the Banach algebra generated by A and B if we do not assume $B \geq 0$.

Example 3.7. *Similarly as in Example 3.6 we can see that operators*

$$A = (x_2 + x_3) \otimes \varphi_2 \quad \text{and} \quad B = x_2 \otimes \varphi_1 + (x_2 - x_1) \otimes \varphi_2 + x_1 \otimes \varphi_3$$

satisfy $AB \geq BA \geq 0$. An easy calculation also shows that $(AB - BA)B(x_2 + 2x_3) = x_2 + 2x_3$, so that 1 is in the spectrum of the operator $(AB - BA)B$.

4. RESULTS

Although the counterexamples in Section 3 answer negatively to [4, Open questions 3.7(2)] in the case of at least three-dimensional lattices, we now provide a positive result in the two-dimensional case.

Proposition 4.1. *Suppose that matrices A and B in $M_2(\mathbb{R})$ satisfy $AB \geq BA \geq 0$. Then either $AB = BA$ or A and B are simultaneously completely decomposable.*

If two matrices commute, then it is well-known that A and B are simultaneously triangularizable. Therefore we can conclude that real matrices $A, B \in M_2(\mathbb{R})$ are simultaneously triangularizable whenever they satisfy the conditions of the preceding proposition. It should be clear that the same result does not hold for $n \geq 3$. If matrices A and B are simultaneously triangularizable, then A and B are upper-triangular with respect to some basis of the space \mathbb{C}^n . Now it is not difficult to see that the matrix $A(AB - BA)$ is nilpotent. However, Example 3.6 shows that there exist matrices $A, B \in M_3(\mathbb{R})$ satisfying $AB \geq BA \geq 0$, and the matrix $A(AB - BA)$ is not nilpotent.

Proof. We may assume $B \neq 0$, as otherwise A and B commute. Suppose first that the matrix AB is indecomposable. Then [1, Lemma 8.14], [1, Corollary 8.23] and the fact that we have $r(AB) = r(BA)$ imply $AB = BA$.

Assume now that AB is decomposable. The standard subspace invariant under AB is invariant under BA as well, since $AB \geq BA \geq 0$. The fact that the space is two-dimensional implies that AB and BA are simultaneously similar to upper-triangular matrices and that this similarity can be obtained by a permutation matrix. We may obviously assume that AB and BA are already upper-triangular. Since $AB \geq BA \geq 0$ and the spectra of upper-triangular matrices are contained on their diagonals, we see that the matrices AB and BA have the same diagonal. Let us write

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}.$$

From

$$AB = \begin{bmatrix} \alpha & \gamma \\ 0 & \beta \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} \alpha & \delta \\ 0 & \beta \end{bmatrix}$$

we obtain $\alpha = ae + bg = ae + cf$, $\beta = cf + dh = bg + dh$ and $ce + dg = ag + ch = 0$.

Let us first consider the case $g = 0$. Then $ce = cf = ch = 0$. If $c = 0$, then A and B are upper-triangular. If $c \neq 0$, then $e = f = h = 0$ and $B = 0$ which is impossible due to our assumption.

Assume now that $g \neq 0$. Then we have $b = \frac{fc}{g}$, $d = -\frac{ce}{g}$ and $a = -\frac{ch}{g}$. Now, a direct calculation shows $\gamma = af + bh = 0$ and $\delta = eb + fd = 0$. Therefore $AB = BA$. \square

Corollary 4.2. *Suppose that matrices A and B in $M_2(\mathbb{R})$ satisfy $AB \geq BA \geq 0$. Then $AB - BA$ is in the radical of the Banach algebra generated by A and B .*

Proof. If $AB = BA$, then there is nothing to prove. Otherwise, Proposition 4.1 implies that A and B are simultaneously completely decomposable. By McCoy's theorem [15, Theorem 1.3.4] the matrix $p(A, B)(AB - BA)$ is nilpotent for every polynomial p in two non-commuting variables. Since the spectral radius is continuous on the set of all $n \times n$ matrices, a matrix $C(AB - BA)$ is nilpotent for every C in the Banach algebra generated by A and B which finishes the proof. \square

Although real matrices A and B satisfying $AB \geq BA \geq 0$ are not triangularizable in general, the situation becomes completely different if A and B are assumed to be positive.

Proposition 4.3. *Let A and B be positive power-compact operators on a complex Banach lattice E with a positive commutator $AB - BA$. Then A and B are simultaneously triangularizable.*

Proof. Assume that A and B are power-compact. Since a closed subspace is invariant under A and B if and only if it is invariant under A and $A + B$, by [8, Lemma 2.2] we may assume that $0 \leq A \leq B$ as we can replace B with $A + B$. We claim that A and B are simultaneously triangularizable. Let \mathcal{C} be a maximal chain of closed ideals that are invariant under B . Since $0 \leq A \leq B$, every ideal in \mathcal{C} is also invariant under A . For every $\mathcal{J} \in \mathcal{C}$ the power-compact operator $B_{\mathcal{J}}$ is ideal-irreducible. [8, Theorem 1.3] and [8, Corollary 3.3] imply that $A_{\mathcal{J}}$ and $B_{\mathcal{J}}$ commute on $\mathcal{J}/\mathcal{J}_-$. Let \mathcal{C}' be a maximal chain of closed subspaces invariant under A and B which contains the chain \mathcal{C} .

We claim that \mathcal{C}' is a triangularizing chain for operators A and B . Since the chain \mathcal{C}' is maximal as a chain of subspaces invariant under A and B , (i) and (ii) in [15, Theorem 7.1.9] are satisfied. Therefore we only need to prove that the dimension of $\mathcal{M}/\mathcal{M}_-$ is at most one for an arbitrary subspace \mathcal{M} in the chain \mathcal{C}' .

Choose an arbitrary subspace $\mathcal{M} \in \mathcal{C}'$. We claim that there exists a closed ideal \mathcal{J} in \mathcal{C} satisfying $\mathcal{J}_- \subseteq \mathcal{M}_- \subseteq \mathcal{M} \subseteq \mathcal{J}$ where \mathcal{J}_- and \mathcal{M}_- are predecessors of \mathcal{J} and \mathcal{M} in chains \mathcal{C} and \mathcal{C}' , respectively. Let \mathcal{J} be the intersection of all closed ideals in \mathcal{C} that contain \mathcal{M} . Let \mathcal{J}_- and \mathcal{M}_- be the predecessors of \mathcal{J} and \mathcal{M} in the chains \mathcal{C} and \mathcal{C}' , respectively. Assume first that $\mathcal{J}_- = \mathcal{J}$. If \mathcal{I} is an arbitrary closed ideal in \mathcal{C} that is properly contained in \mathcal{J} , then \mathcal{I} is contained in \mathcal{M} , so that \mathcal{J}_- is also contained in \mathcal{M} . Since $\mathcal{M} \subseteq \mathcal{J}$ and $\mathcal{J}_- \subseteq \mathcal{M}$, we have $\mathcal{J} = \mathcal{J}_- = \mathcal{M} = \mathcal{M}_-$. Assume now that \mathcal{J}_- is a proper subset of \mathcal{J} . Since \mathcal{C}' is a chain, then \mathcal{J}_- is a subset of \mathcal{M} , as \mathcal{J} is the intersection of all closed ideals of \mathcal{C} that contain \mathcal{M} , and \mathcal{J}_- is a closed linear span of all closed ideals of \mathcal{C} that are contained in \mathcal{M} . If $\mathcal{J}_- = \mathcal{M}$, then \mathcal{J}_- is actually equal to \mathcal{J} which is impossible due to our assumption $\mathcal{J}_- \neq \mathcal{J}$. Therefore \mathcal{J}_- is a proper subset of \mathcal{M} which implies $\mathcal{J}_- \subseteq \mathcal{M}_-$ so that we have $\mathcal{J}_- \subseteq \mathcal{M}_- \subseteq \mathcal{M} \subseteq \mathcal{J}$ and the claim is proved.

Assume now that the dimension of $\mathcal{M}/\mathcal{M}_-$ is at least 2. Since $A_{\mathcal{J}}$ and $B_{\mathcal{J}}$ commute on $\mathcal{J}/\mathcal{J}_-$ and $\mathcal{J}_- \subseteq \mathcal{M}_-$, the induced operators $A_{\mathcal{M}}$ and $B_{\mathcal{M}}$ of the operators A and B on the quotient Banach space $\mathcal{M}/\mathcal{M}_-$ also commute. Suppose first that at least one of $A_{\mathcal{M}}$ and $B_{\mathcal{M}}$ is not a scalar multiple of the identity operator. Without any loss of generality assume that $A_{\mathcal{M}}$ is not a scalar multiple of the identity operator. If $A_{\mathcal{M}}$ is nilpotent, then its kernel is invariant under both $A_{\mathcal{M}}$ and $B_{\mathcal{M}}$, so that the preimage of the kernel of the operator $A_{\mathcal{M}}$ is a closed subspace invariant under A and B that is properly contained between \mathcal{M}_- and \mathcal{M} . This contradicts the maximality of \mathcal{C}' . If $A_{\mathcal{M}}$ is not nilpotent, then some power $(A_{\mathcal{M}})^n$ is a nonzero compact operator. By Lomonosov's theorem [12] there exists a nontrivial closed subspace of $\mathcal{M}/\mathcal{M}_-$ invariant under $A_{\mathcal{M}}$ and $B_{\mathcal{M}}$. Similarly as above this is again a contradiction with maximality of \mathcal{C}' . If both $A_{\mathcal{M}}$ and $B_{\mathcal{M}}$ are scalar multiples of the identity operator, then every subspace of $\mathcal{M}/\mathcal{M}_-$ is invariant under $A_{\mathcal{M}}$ and $B_{\mathcal{M}}$ which would (similarly as above) lead to a contradiction with the maximality of the chain \mathcal{C}' .

Therefore the dimension of $\mathcal{M}/\mathcal{M}_-$ is less than or equal to one and the chain \mathcal{C}' is a triangularizing chain for the operators A and B . \square

In the proof of the preceding Proposition we applied the Lomonosov theorem to prove that the chain \mathcal{C}' is a triangularizing chain for operators A and B . In the real case this cannot be done always. If the dimension of $\mathcal{M}/\mathcal{M}_-$ is finite and at least three, then we can proceed as in the proof of Proposition 4.3 also in the real case, since every real matrix on \mathbb{R}^n with n at least three has a nontrivial hyperinvariant subspace, which leads to a contradiction with the maximality of the chain \mathcal{C}' . However, the problem occurs when the dimension of $\mathcal{M}/\mathcal{M}_-$ is precisely two. In the following example we construct a positive indecomposable matrix $A \in \mathbb{R}^3$ with a two dimensional invariant subspace \mathcal{M} such that the restriction of A to \mathcal{M} is irreducible.

Example 4.4. Take the indecomposable matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and \mathcal{M} the vector space spanned by $\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. A straightforward calculations show that subspace \mathcal{M} is invariant under A , and the restriction of A to \mathcal{M} is irreducible, since the only (real) eigenspace of A is spanned by the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ which does not belong to \mathcal{M} .

It should be noted that every positive matrix on \mathbb{R}^2 is triangularizable since the spectral radius of a positive matrix is always in its spectrum.

The following theorem is the main result of this paper. It provides a positive answer to the remaining part of [4, Open questions 3.7(1)] and at the same time it tells us actually that the commutator $AB - BA$ of a positive compact operator and a positive operator is contained in the radical of a bigger Banach algebra which contains the Banach algebra generated by A and B .

Theorem 4.5. Let A be a positive compact operator on a Banach lattice E and let \mathcal{A}_1 and \mathcal{A}_2 be the Banach algebras generated by $\langle A \rangle$ and $[A]$, respectively.

- (a) The operator $AB - BA$ is in the radical of the Banach algebra \mathcal{A}_1 for every $B \in \mathcal{A}_1$.
- (b) The operator $AB - BA$ is in the radical of the Banach algebra \mathcal{A}_2 for every $B \in \mathcal{A}_2$.

Proof. Let \mathcal{A} be the algebra generated by $\langle A \rangle$ and let us choose B and C from \mathcal{A} . The operators B and C are finite linear combinations of words in letters from $\langle A \rangle$. Let B_1, \dots, B_n and C_1, \dots, C_m be operators in $\langle A \rangle$ which occur as letters in words whose appropriate linear combinations are equal to B and C , respectively. The operator $D := A + B_1 + \dots + B_n + C_1 + \dots + C_m$ satisfies $0 \leq A \leq D$ and the commutator

$$AD - DA = \sum_{i=1}^n (AB_i - B_iA) + \sum_{j=1}^m (AC_j - C_jA)$$

is obviously a positive operator on E . Let \mathcal{C} be a maximal chain of closed ideals invariant under D . Maximality of the chain \mathcal{C} implies that \mathcal{C} is complete and that for every $\mathcal{J} \in \mathcal{C}$ with $\dim(\mathcal{J}/\mathcal{J}_-) > 1$ the operator $D_{\mathcal{J}}$ is ideal-irreducible on $\mathcal{J}/\mathcal{J}_-$. Also, every closed ideal in \mathcal{C} is invariant under A , B_i and C_j for all $1 \leq i \leq n$ and $1 \leq j \leq m$, so that it is also invariant under A , B and C . [10, Corollary 3.5] implies that $A_{\mathcal{J}}$ and $D_{\mathcal{J}}$ commute on $\mathcal{J}/\mathcal{J}_-$. This implies, in particular, that $A_{\mathcal{J}}$ commutes with $(B_i)_{\mathcal{J}}$ for all $1 \leq i \leq n$, so that actually $A_{\mathcal{J}}$ and $B_{\mathcal{J}}$ commute on $\mathcal{J}/\mathcal{J}_-$. By Ringrose's theorem for complete chains [15, Theorem 7.2.7], it follows that

$$\sigma(C(AB - BA)) = \bigcup_{\substack{\mathcal{J} \in \mathcal{C} \\ \mathcal{J} \neq \mathcal{J}_-}} \sigma(C_{\mathcal{J}}(A_{\mathcal{J}}B_{\mathcal{J}} - B_{\mathcal{J}}A_{\mathcal{J}})) \cup \{0\} = \{0\}.$$

If B and C in \mathcal{A}_1 are arbitrary, then there exist sequences of operators $\{B_n\}_{n \in \mathbb{N}}$ and $\{C_n\}_{n \in \mathbb{N}}$ in \mathcal{A} converging to B and C , respectively, so that $C_n(AB_n - B_nA)$ converges to $C(AB - BA)$. Since the norm limit of compact quasi-nilpotent operators is quasi-nilpotent, $C(AB - BA)$ is quasi-nilpotent which finishes the proof of (a). The proof of (b) is similar so we omit it. □

Corollary 4.6. *Suppose that A and B be positive operators on a Banach lattice E with a positive commutator $AB - BA$. If at least one of the operators A and B is compact, then $AB - BA$ is contained in the radical of the Banach algebra generated by A and B .*

Proof. Suppose that A is a compact operator. Let us denote by \mathcal{A} the Banach algebra generated by A and B , and let us denote by \mathcal{A}_1 the Banach algebra generated by $\langle A \rangle$. Since $AB - BA$ is in the radical of \mathcal{A}_1 by Theorem 4.5(a), it follows that $r(C(AB - BA)) = 0$ for an arbitrary operator $C \in \mathcal{A}_1$. The fact that we have $\mathcal{A} \subseteq \mathcal{A}_1$ implies $r(C(AB - BA)) = 0$

for every $C \in \mathcal{A}$ which means that $AB - BA$ is in the radical of \mathcal{A} . For the proof when B is compact we repeat the proof above and apply Theorem 4.5(b). \square

In the case of power-compact operators we can obtain the following result.

Theorem 4.7. *If A and B are positive power-compact operators on a Banach lattice with a positive commutator $AB - BA$, then some power of $AB - BA$ is in the radical of the Banach algebra generated by A and B .*

Proof. Let \mathcal{A} be the Banach algebra generated by A and B . By [8, Lemma 2.3], the operator $AB - BA$ is a power-compact operator. If $AB - BA$ is nilpotent, then it is clear that the appropriate power of the operator $AB - BA$ is in the radical of \mathcal{A} . Therefore we may assume that $AB - BA$ is not nilpotent. There exists a positive integer n such that $(AB - BA)^n$ is a compact operator.

Suppose first that the lattice E is a complex Banach lattice. Proposition 4.3 implies that A and B are simultaneously triangularizable. Let \mathcal{C} be one of the triangularizing chains for operators A and B . It is not hard to see that \mathcal{C} is a triangularizing chain for every operator C from \mathcal{A} . Since every diagonal coefficient of the operator $AB - BA$ is zero, every diagonal coefficient of the operator $C(AB - BA)^n$ is zero as well. Compactness of the operator $(AB - BA)^n$ and [15, Theorem 7.2.3] implies that $C(AB - BA)^n$ is quasi-nilpotent, and since $C \in \mathcal{A}$ was arbitrary we obtain that $(AB - BA)^n$ is contained in the radical of \mathcal{A} .

Assume now that E is a real Banach lattice. Let $E_{\mathbb{C}}$ be the complexification of the Banach lattice E and let $A_{\mathbb{C}}$ and $B_{\mathbb{C}}$ be the standard complexifications of operators A and B on $E_{\mathbb{C}}$, respectively. It is not hard to see that we have $A_{\mathbb{C}}B_{\mathbb{C}} \geq B_{\mathbb{C}}A_{\mathbb{C}}$ on $E_{\mathbb{C}}$, so that by Proposition 4.3 operators $A_{\mathbb{C}}$ and $B_{\mathbb{C}}$ are simultaneously triangularizable. Since we have

$$(A_{\mathbb{C}}B_{\mathbb{C}} - B_{\mathbb{C}}A_{\mathbb{C}})^n = ((AB - BA)_{\mathbb{C}})^n = ((AB - BA)^n)_{\mathbb{C}},$$

the operator $(A_{\mathbb{C}}B_{\mathbb{C}} - B_{\mathbb{C}}A_{\mathbb{C}})^n$ is a nonzero compact operator on $E_{\mathbb{C}}$. By the proof above we see that $(A_{\mathbb{C}}B_{\mathbb{C}} - B_{\mathbb{C}}A_{\mathbb{C}})^n$ is contained in the radical of the algebra $\mathcal{A}_{\mathbb{C}}$ which implies that $r(C_{\mathbb{C}}(A_{\mathbb{C}}B_{\mathbb{C}} - B_{\mathbb{C}}A_{\mathbb{C}})^n) = 0$ for all $C_{\mathbb{C}} \in \mathcal{A}_{\mathbb{C}}$. This implies that $r(C(AB - BA)^n) = 0$ for all $C \in \mathcal{A}$ finishing the proof. \square

5. A RESULT FOR THE VOLTERRA OPERATOR AND THE DONOGHUE OPERATOR

An operator on a Banach space is said to be *unicellular* if its lattice of closed invariant subspaces is totally ordered. An application of the Lomonosov theorem shows that every compact operator is triangularizable which implies that every unicellular compact operator has a unique triangularizing chain. The Volterra operator acting on $L^2[0, 1]$ defined by $(Vf)(x) = \int_0^x f(t)dt$ is a well-known example of a unicellular operator [14] with its lattice of closed invariant subspaces consisting of subspaces of the form $L^2[t, 1]$, i.e., of all functions from $L^2[0, 1]$ which are zero almost everywhere on the interval $[0, t]$. Volterra operator has a lot of interesting properties. For an example, if a normal operator N commutes with a Volterra operator, then N is a multiple of the identity operator [9]. An operator T on a Banach space X is said to be a *strong quasi-affinity* whenever for arbitrary closed subspace \mathcal{M} invariant under T we have that $T\mathcal{M}$ is dense in \mathcal{M} .

The Volterra operator has this remarkable property. It is not hard to see that the range of V is dense in $L^2[0, 1]$. The function $x \mapsto \frac{x-t}{1-t}$ induces a topological isomorphism $\Phi : L^2[0, 1] \rightarrow L^2[t, 1]$ defined by

$$(\Phi f)(x) = f\left(\frac{x-t}{1-t}\right).$$

Let us denote by V_t the restriction of the Volterra operator to its invariant subspace $L^2[t, 1]$. It is not hard to see that V_t satisfies

$$(V_t \Phi f)(s) = (1-t) \int_0^{\frac{s-t}{1-t}} f(v)dv = (1-t)(\Phi V f)(s)$$

for all $s \in [t, 1]$. Since V has a dense range, V_t also has a dense range.

The other example of a unicellular operator is the Donoghue operator on a separable Hilbert space \mathcal{H} . Let $\{e_n\}_{n \in \mathbb{N}_0}$ be an orthonormal basis of \mathcal{H} and let $\{w_n\}_{n \in \mathbb{N}}$ be a sequence of nonzero complex numbers such that $\{|w_n|\}_{n \in \mathbb{N}}$ is monotone decreasing and is in the space l^2 . The *Donoghue operator* S with a weight sequence $\{w_n\}_{n \in \mathbb{N}}$ is defined by $Se_0 = 0$ and $Se_n = w_n e_{n-1}$ for $n > 0$. The operator S is a unicellular operator [14] with its lattice of closed invariant subspaces equal to $\{\mathcal{M}_n\}_{n \in \mathbb{N}_0}$ where \mathcal{M}_n is the linear span of the vectors e_0, \dots, e_n . An easy calculation shows that $S\mathcal{M}_n = \mathcal{M}_{n-1}$, so that the induced operator by the operator S on the quotient Banach space $\mathcal{M}_n/\mathcal{M}_m$ is nonzero whenever the dimension of $\mathcal{M}_n/\mathcal{M}_m$ is at least 2. The later fact will be considered in

general in Proposition 5.1. We will be only concerned with the positive (the weights are positive) Donoghue operator acting on l^2 ordered componentwise.

It is well-known that the Volterra operator and the Donoghue operator are compact operators.

Proposition 5.1. *Let X be a Banach space and let T be a compact unicellular operator on X . Then for an arbitrary subspace \mathcal{M} from the triangularizing chain \mathcal{C} of T only one of the following statements hold.*

- (a) $\mathcal{M} = \overline{T\mathcal{M}}$.
- (b) $\mathcal{M} = \mathcal{M}_-$ and $\overline{T\mathcal{M}}$ is of codimension 1 in \mathcal{M} .
- (c) $\overline{T\mathcal{M}} = \mathcal{M}_-$ is of codimension 1 in \mathcal{M} .

Moreover, if \mathcal{C} is a continuous chain, then T is a strong quasi-affinity.

It should be noted that Proposition 5.1 immediately implies that V is a strong quasi-affinity and that V is quasi-nilpotent by [15, Example 7.2.5].

Proof. Assume that $\overline{T\mathcal{M}}$ is a proper subspace of \mathcal{M} . We claim that the dimension of the quotient space $\mathcal{M}/\overline{T\mathcal{M}}$ is at most one. Otherwise, one can find two proper incomparable closed subspaces \mathcal{N}_1 and \mathcal{N}_2 of \mathcal{M} that properly contain $\overline{T\mathcal{M}}$. Since \mathcal{N}_1 and \mathcal{N}_2 are also invariant under T , this contradicts unicellularity of T . Unicellularity of T also implies that $\overline{T\mathcal{M}} \in \mathcal{C}$. Since the codimension of \mathcal{M}_- in \mathcal{M} is also at most one, we have $\mathcal{M}_- = \overline{T\mathcal{M}}$ or $\mathcal{M}_- = \mathcal{M}$.

If \mathcal{C} is a continuous chain, then it is obvious that (a) holds. □

Proposition 5.2. *Let E be a Banach lattice and let A be a positive compact unicellular operator on E such that the unique triangularizing chain for T consists of closed ideals of E . If a positive operator B semi-commutes with A , then A and B are simultaneously triangularizable.*

Proof. Let \mathcal{C} be a maximal chain of closed subspaces invariant under both A and B . By the assumption, every subspace in \mathcal{C} is an ideal of E . Due to the maximality, \mathcal{C} is a complete chain. Suppose now that the dimension of $\mathcal{J}/\mathcal{J}_-$ is at least two for some $\mathcal{J} \in \mathcal{C}$. Since the operator $A_{\mathcal{J}} + B_{\mathcal{J}}$ is ideal-irreducible on $\mathcal{J}/\mathcal{J}_-$ and semi-commutes with the

operator $A_{\mathcal{J}}$, [10, Corollary 3.5] implies

$$(A_{\mathcal{J}} + B_{\mathcal{J}})A_{\mathcal{J}} = A_{\mathcal{J}}(A_{\mathcal{J}} + B_{\mathcal{J}}),$$

so that we have $A_{\mathcal{J}}B_{\mathcal{J}} = B_{\mathcal{J}}A_{\mathcal{J}}$. We claim that $A_{\mathcal{J}}$ is nonzero operator on $\mathcal{J}/\mathcal{J}_-$. Otherwise we have $A\mathcal{J} \subseteq \mathcal{J}_-$, and since Proposition 5.1 implies that the codimension of $\overline{A\mathcal{J}}$ is at most one, we have that the dimension of $\mathcal{J}/\mathcal{J}_-$ is at most one as well and we reached to a contradiction.

The operator $A_{\mathcal{J}}$ is not a scalar multiple of the identity operator on $\mathcal{J}/\mathcal{J}_-$ as otherwise A would have at least two incomparable closed invariant subspaces which contradicts its unicellularity. By [17, Corollary 2.4] (if E is a real lattice) or [12] (if E is a complex lattice), the operator $A_{\mathcal{J}}$ has a nontrivial hyperinvariant subspace. However, this is in contradiction with the maximality of the chain \mathcal{C} . Therefore, \mathcal{C} is a maximal chain of closed subspaces of E . \square

Whenever one of the operators A and B in Proposition 4.3 is the Volterra operator or a Donoghue operator, then the other operator does not need to have any compactness properties.

Corollary 5.3. *Let V be the Volterra operator on $L^2[0, 1]$ and let S be the Donoghue operator on l^2 . Then the following statements hold.*

- (a) *If a positive operator T on $L^2[0, 1]$ semi-commutes with V then V and T are simultaneously ideal-triangularizable.*
- (b) *If a positive operator T on l^2 semi-commutes with S , then S and T are simultaneously ideal-triangularizable.*

In [4, Example 3.3], a Banach algebra generated by the Volterra operator V and a multiplication operator M on $L^2[0, 1]$ defined by $(Mf)(x) = xf(x)$ was considered. Since the relation $MV - VM = V^2$ implies $MV \geq VM \geq 0$, Corollary 4.6 implies that $MV - VM$ is contained in the radical of the Banach algebra generated by V and M . The authors [4] observed that this actually follows from the fact that V and M are simultaneously ideal-triangularizable with the unique triangularizing chain $\{L^2[t, 1]\}_{t \in [0, 1]}$. The following result states that the Volterra operator is actually in the radical of the possibly much bigger Banach algebra generated by $\langle V \rangle$ and $[V]$.

Theorem 5.4. *Let V be the Volterra operator on $L^2[0, 1]$ and let S be the Donoghue operator on l^2 . Then V is in the radical of the Banach algebra generated by $\langle V \rangle$ and $[V]$, and S is in the radical of the Banach algebra generated by $\langle S \rangle$ and $[S]$.*

Proof. By Corollary 5.3 the set $\langle V \rangle \cup [V]$ is triangularizable, so that the closure of the algebra \mathcal{A} generated $\langle V \rangle$ and $[V]$ is triangularizable as well. [15, Theorem 7.2.4] implies that the operator AV is quasi-nilpotent for all $A \in \mathcal{A}$ which finishes the proof.

For the proof in the case of the Donoghue operator note first that it is quasi-nilpotent by [11, Problem 96] and its solution. [15, Theorem 7.2.3] implies that every diagonal coefficient of the operator S is zero. Since $\langle S \rangle \cup [S]$ is triangularizable by Corollary 5.3, the Banach algebra \mathcal{B} generated by $\langle S \rangle \cup [S]$ is triangularizable as well. For all $B \in \mathcal{B}$ every diagonal coefficient of the operator BS is zero, so that the operator BS is quasi-nilpotent by [15, Theorem 7.2.3] and the proof is finished. \square

Remark 5.5. *Let V be the Volterra operator and let \mathcal{A} be the Banach algebra generated by all compact operators in $\langle V \rangle \cup [V]$. Then \mathcal{A} is triangularizable with the unique triangularizing chain $\{L^2[t, 1]\}_{t \in [0, 1]}$. By [15, Corollary 7.2.4] every operator in \mathcal{A} is quasi-nilpotent, so that \mathcal{A} is a radical Banach algebra.*

Remark 5.6. *Suppose that A and B be positive operators on a Banach lattice E with a positive commutator $AB - BA$. If one of the operators is power compact, then $AB - BA$ is quasi-nilpotent by [5, Corollary 3.2]. In particular, this implies that a positive commutator $AB - BA$ of positive power compact operators A and B is quasi-nilpotent which also follows from Theorem 4.7.*

We finish this paper with two open questions.

- (a) How big is the Banach algebra generated by $\langle V \rangle \cup [V]$?
- (b) Can we assume in Theorem 4.7 that only one of the operators A and B is power compact?

Acknowledgments. This work was supported in part by the Slovenian Research Agency.

REFERENCES

- [1] Y. A. Abramovich, C. D. Aliprantis, *An invitation to operator theory*, American Mathematical Society, Providence, 2002.
- [2] C.D. Aliprantis, O. Burkinshaw, *Positive operators*, reprint of the 1985 original, Springer, Dordrecht, 2006.
- [3] F. F. Bonsall, J. Duncan, *Complete normed algebras*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 80. Springer-Verlag, New York-Heidelberg, 1973.
- [4] J. Bračič, R. Drnovšek, Y. B. Farforovskaya, E. L. Rabkin, J. Zemánek, *On positive commutators*, Positivity **14** (2010), 431–439.
- [5] R. Drnovšek, *Once more on positive commutators*, Studia Math. **211** (2012), 241–245.
- [6] R. Drnovšek, *Triangularizing semigroups of positive operators on an atomic normed Riesz spaces*, Proc. Edin. Math. Soc. **43** (2000), 43–55.
- [7] R. Drnovšek, M. Kandić, *Ideal-triangularizability of semigroups of positive operators*, Integral Equations and Operator Theory **64** (2009), 539–552.
- [8] R. Drnovšek, M. Kandić, *More on positive commutators*, J. Math. Anal. Appl. **373** (2011), 580–584.
- [9] A. F. M. ter Elst, M. Sauter, J. Zemánek, *Generation and commutation properties of the Volterra operator*, Arch. Math. **99** (2012), no. 5, 467–479.
- [10] N. Gao, *On commuting and semi-commuting positive operators*, Proc. Amer. Math. Soc., in press.
- [11] P. Halmos, *A Hilbert space problem book*, Second edition, Graduate Texts in Mathematics, Springer-Verlag, New York-Berlin, 1982.
- [12] V. Lomonosov, *Invariant subspaces of the family of operators that commute with a completely continuous operator*. (Russian) Funkcional. Anal. i Priložen. **7** (1973), no. 3, 55–56.
- [13] W. A. J. Luxemburg, A. C. Zaanen, *Riesz spaces I*, North Holland, Amsterdam, 1971.
- [14] H. Radjavi, P. Rosenthal, *Invariant subspaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 77. Springer-Verlag, New York-Heidelberg, 1973.
- [15] H. Radjavi, P. Rosenthal, *Simultaneous Triangularization*, Springer-Verlag, New York, 2000.
- [16] H. H. Schaefer, *Banach lattices*, Springer-Verlag, Berlin-Heidelberg-New York, 1974.
- [17] G. Sirotkin, *A Version of The Lomonosov Subspace Theorem for Real Banach Spaces* Indiana Univ. Math. J. **54** (2005), no. 1, 257–262.
- [18] A.C. Zaanen, *Introduction to Operator Theory in Riesz Spaces*, Springer-Verlag Berlin Heidelberg, 1997.

Marko Kandić, Klemen Šivic:

Faculty of Mathematics and Physics

University of Ljubljana

Jadranska 19

1000 Ljubljana

Slovenia

E-mails : marko.kandic@fmf.uni-lj.si, klemen.sivic@fmf.uni-lj.si